PHYS 301 Tutorial #5 Sol'ns Oct. 4, 2024

1. By Gauss's law, the electric field inside the shell is zero. Outside the shell, $\vec{E}$ looks like that of a pt. charge:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\therefore E^2 = \vec{E}\cdot\vec{E} = \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4} \quad \text{for } r > R.$$

$$\therefore W = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\\text{space}}} E^2 \, d\tau$$

$$= \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=R}^{\infty} \frac{1}{r^4} r^2 dr$$

since $E = 0$ for $r < R$

$$= \frac{4\pi\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_{r=R}^{\infty} \frac{1}{r^2}\,dr$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_R^{\infty} = \frac{q^2}{8\pi\varepsilon_0}\left[-\frac{1}{\infty} + \frac{1}{R}\right]$$

$$\boxed{\therefore W = \frac{q^2}{8\pi\varepsilon_0 R}}$$

Same result as obtained in class.

2. For pt. charges

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)}_{\substack{\text{potential at } q_i \\ \text{due to all other} \\ \text{charges.}}}$$

$$\therefore W = \frac{1}{2}\sum_{i=1}^{n} q \sum_{j\neq i}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q(-1)^n}{na}$$

$$= \frac{nq}{2}\frac{q}{4\pi\varepsilon_0 a}\sum_{j\neq i}^{n}\frac{(-1)^n}{n}$$

$j\neq i$ sum is over all charges to the left & right of q_i. Instead, sum over only charges (say the ones to the right) and multiply by 2.

$$\therefore W = \frac{n q^2}{4\pi \varepsilon_0 a} \sum_{j=1}^{\infty} \frac{(-1)^n}{n}$$

Consider $\ln(1+x)$ and Taylor series expand about $x=0$.

$$\ln(1+x) = \ln(1) + \frac{1}{1+x}\bigg|_{x=0} x - \frac{1}{2}\left(\frac{1}{1+x}\right)^2\bigg|_{x=0} x^2$$

$$+ \frac{2}{3!}\left(\frac{1}{1+x}\right)^3\bigg|_{x=0} x^3 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Subbing in $x=1$, gives

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots = -\sum_{j=1}^{\infty} \frac{(-1)^n}{n}$$

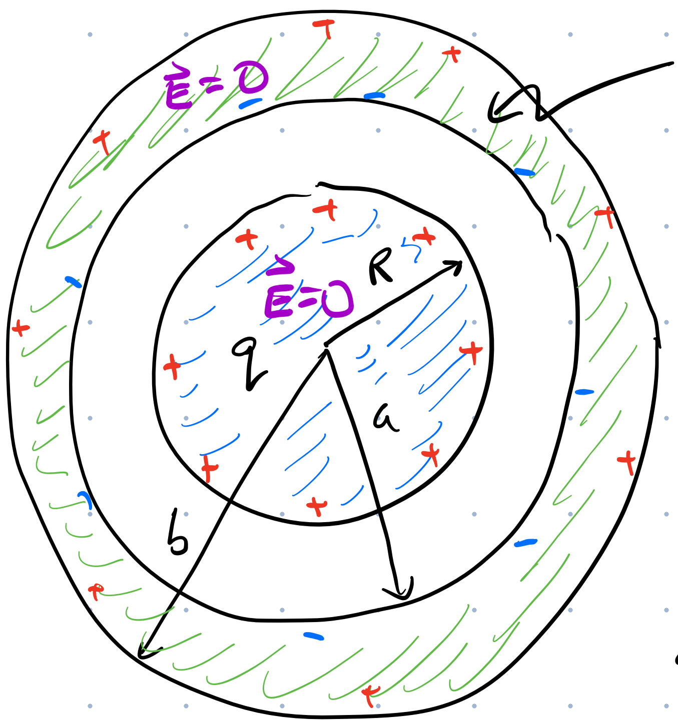
$$\therefore \sum_{j=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2$$

Finally, $W = \ln 2 \dfrac{n q^2}{4\pi \varepsilon_0 a}$ but $n \to \infty$

∴ Work per particle is

$$\boxed{\frac{W}{n} = \ln 2 \frac{q^2}{4\pi\varepsilon_0 a} \Rightarrow \alpha = \ln 2}$$

3.



$Q_{shell} = Q_{inner} + Q_{outer}$

$= 0$

$\therefore Q_{inner} = -Q_{outer}$

Because sphere & shell are both conductors, all charge is on surfaces.

(a) q is on surface of sphere.

$$\boxed{\therefore \sigma_R = \frac{q}{4\pi R^2}}$$

A Gaussian surface placed just outside $r=a$, must enclose zero charge since $E=0$.

$$\therefore \; q_a + q = 0 \qquad \therefore \; q_a = -q.$$

$$\boxed{\therefore \; \sigma_a = \frac{-q}{4\pi a^2}}$$

A Gaussian surface place just outside $r=b$ encloses charge $Q_{encl.} = q + q_a + q_b = q$

$$\therefore \; q_b = -q_a = +q.$$

$$\boxed{\therefore \; \sigma_b = \frac{q}{4\pi b^2}}$$

(b) $$\Delta V = -\int \vec{E}\cdot d\vec{l}$$

$$\cancelto{0}{V(\infty)} - V(0) = -\int_0^\infty \vec{E}\cdot d\vec{l}.$$

$$0 < r < R : \vec{E} = 0$$

$$a < r < b : \vec{E} = 0$$

$$\therefore V(0) = \int_R^a \vec{E}\cdot d\vec{l} + \int_b^\infty \vec{E}\cdot d\vec{l}$$

In both regions $R < r < a$ & $b < r < \infty$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r} \Rightarrow \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}dr$$

$$\therefore V(0) = \frac{q}{4\pi\varepsilon_0}\left[\int_R^a \frac{dr}{r^2} + \int_b^\infty \frac{dr}{r^2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\Big|_R^a - \frac{1}{r}\Big|_b^\infty \right]$$

$$\boxed{\therefore V(0) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{a} + \frac{1}{b} \right]}$$

(c) If $Q_{outer} = 0$, nothing changes for Q_{inner} or q.

$$\boxed{\therefore \sigma_R = \frac{q}{4\pi R^2} \quad \sigma_a = \frac{-q}{4\pi a^2} \quad \sigma_b = 0}$$

For $\vec{E}$, $Q_{encl} = 0$ for a Gaussian surface just outside $r = b$.

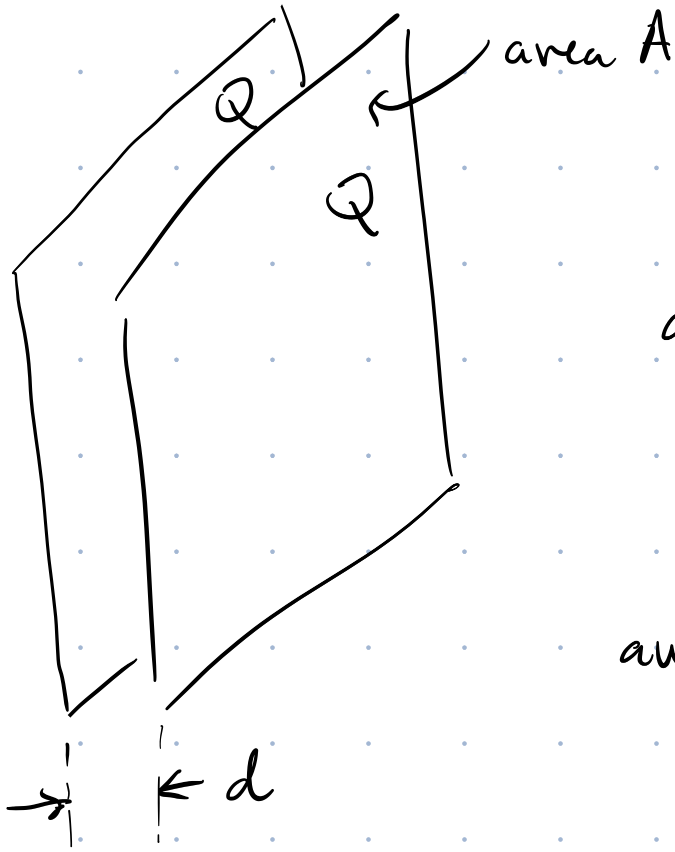
$$\therefore \Delta V = \cancel{V(\infty)}^{\,0} - V(0) = -\int_0^\infty \vec{E} \cdot d\vec{l}$$

$$\therefore V(0) = \int_R^a \frac{q}{4\pi\varepsilon_0} \frac{dr}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0} \left(-\frac{1}{r} \Big|_R^a \right)$$

$$\boxed{\therefore V(0) = \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{R} - \frac{1}{a} \right)}$$

4.



Each plate produces an electric field

$$E = \frac{\sigma}{2\varepsilon_0}$$ pointing away from its surfaces.

$\therefore$ By the superposition principle, $\vec{E} = 0$ between the plates and $E = \frac{\sigma}{\varepsilon_0}$ outside the plates.

$$\therefore P = \frac{\varepsilon_0}{2}E^2 = \frac{\varepsilon_0}{2}\frac{\sigma^2}{\varepsilon_0^2} = \boxed{\frac{Q^2}{2\varepsilon_0 A^2}}$$

5. Gauss's Law in differential form is

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{E} = \frac{k}{r}\left[3\hat{r} + 2\sin\theta\cos\theta\sin\phi\,\hat{\theta} + \sin\theta\cos\phi\,\hat{\phi}\right]$$

$$\therefore \rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$$

In spherical coordinates

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, E_\theta\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$\frac{\partial}{\partial r}\left(r^2 E_r\right) = k\frac{\partial}{\partial r}\left(\frac{3r^2}{r}\right) = 3k$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\, E_\theta\right) = \frac{\partial}{\partial\theta}\left(\frac{2k}{r}\sin^2\theta\cos\theta\sin\phi\right)$$

$$= \frac{4k}{r}\sin\theta\cos^2\theta\sin\phi - \frac{2k}{r}\sin^3\theta\sin\phi$$

$$= \frac{2k}{r}\sin\theta\sin\phi\left(2\cos^2\theta - \sin^2\theta\right)$$

$$\frac{\partial E_\phi}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{k}{r}\sin\theta\cos\phi\right) = -\frac{k}{r}\sin\theta\sin\phi$$

$$\therefore\ \vec{\nabla}\cdot\vec{E} = \frac{3k}{r^2} + \frac{2k}{r^2}\sin\phi\left(2\cos^2\theta - \sin^2\theta\right) - \frac{k}{r^2}\sin\phi$$

$$= \frac{k}{r^2}\left[3 + \sin\phi\left(4\cos^2\theta - 2\sin^2\theta - 1\right)\right]$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\therefore\ 4\cos^2\theta - 2\sin^2\theta - 1$$

$$= 4\cos^2\theta - 2\left(1 - \cos^2\theta\right) - 1$$

$$= 6\cos^2\theta - 3$$

$$= 3(\underbrace{2\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)} - 1)$$

$$= 3\left(1 + \cos 2\theta - 1\right) = 3\cos 2\theta$$

$$\therefore \vec{\nabla} \cdot \vec{E} = \frac{k}{r^2}\left[3 + 3\cos 2\theta \sin \phi\right]$$

$$\boxed{\therefore \rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} = \frac{3\varepsilon_0 k}{r^2}\left[1 + \cos 2\theta \sin \phi\right]}$$